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ON THE "BEST" FIRST-ORDER LINEAR SHELL THEORY

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INTRODUCTION

In marked contrast to the theories of bending and stretching of flat plates, the general linear theory of thin elastic shells has not yet received a universally accepted formulation; it is only a slight exaggeration to say that each investigator favors a different theory. The comparative studies of Koiter [1960] clarify and substantiate the widely held impression that there is little difference, from the point of view of accuracy, among many of the existing sets of shell equations, and also show that some must be regarded with suspicion. Nevertheless, it is inconvenient to have to study and assess the equations underlying each new work on shells.

A more satisfactory state of affairs would prevail if a set of equations that uniquely embody certain clearly desirable characteristics could be logically deduced and then generally adopted. A derivation of this type, leading to equations originally presented in lines-of-curvature coordinates by Sanders [1959], is given in this paper.

A TENSOR THEORY

The first criterion to be met in the search for a standard theory is that it be susceptible to formulation in general tensor notation in

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terms of an arbitrary coordinate system in the middle surface of shells of arbitrary shape. We cannot defend this requirement on strict logical grounds. It is not even clear that exact laws of nature must necessarily be expressible in tensor form, and so it is even less possible to insist on a tensor representation of approximate theories. But the elegance of tensor notation, the generality it affords, and its utility in theoretical studies constitute strong arguments in its favor. At the same time, general tensor equations in arbitrary coordinate systems can, of course, always be transformed into "physical" equations in special coordinate systems; the reverse process is not always possible.

Let the middle surface of the undeformed shell be defined by means of the cartesian position vector $x^i(\xi^\alpha)$. In the present paper the ξ^α coordinates are used to label material particles in both the undeformed and deformed middle surfaces. The metric tensor and the curvature tensor of the undeformed middle surface are given by

$$g_{\alpha\beta} = x^i_{,\alpha} x^i_{,\beta} \tag{1}$$

and

$$b_{\alpha\beta} = -x^i_{,\alpha\beta} N^i \tag{2}$$

where N^i is the unit normal to the undeformed middle surface; commas denote covariant differentiation throughout this paper. According to the fundamental theorem of surfaces these two tensors completely define a surface (within a rigid body motion) provided they satisfy the Gauss and Codazzi identities. The deformed middle surface is defined by its cartesian position vector

$$y^i = x^i + u^\alpha x^i_{,\alpha} + wN^i \tag{3}$$

where u^α and w are the displacements given as functions of ξ^α. The two fundamental tensors of the deformed middle surface, $G_{\alpha\beta}$ and $B_{\alpha\beta}$, are given by formulas similar to (1) and (2).

FIRST ORDER THEORY, THE KIRCHOFF ASSUMPTIONS, AND STRAINS

The meaning of the characterization "first-order" (or "first approximation") in shell theory is that the deformed state of the shell is determined entirely by the deformed configuration of its middle surface. The displaced locations of material points off the middle surface are rendered determinate by means of the Kirchoff assumption: material normals to the undeformed middle surface do not change length and remain normal to the deformed middle surface after deformation of the shell. Accordingly, in any first order theory, the deformed shell is characterized geometrically entirely by $G_{\alpha\beta}$ and $B_{\alpha\beta}$, the metric and curvature tensors of the deformed middle surface.

It follows then, that the distortions of the shell are defined adequately by the <u>stretching strain</u> tensor

$$E_{\alpha\beta} = \frac{1}{2}(G_{\alpha\beta} - g_{\alpha\beta}) \tag{4}$$

and the <u>bending strain</u> tensor

$$K_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} \tag{5}$$

These tensors obviously vanish for rigid body motions. In terms of the surface displacement vector u^α and the normal displacement w,

these strain tensors when linearized, can be calculated to be

$$E_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + b_{\alpha\beta}\, w \tag{6}$$

$$K_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) + \frac{1}{2}(b^{\gamma}_{\alpha}\, u_{\gamma,\beta} + b^{\gamma}_{\beta}\, u_{\gamma,\alpha}) + b^{\gamma}_{\alpha}\, b_{\gamma\beta}\, w \tag{7}$$

where the rotation vector ϕ_α is

$$\phi_\alpha = -w_{,\alpha} + b_{\alpha\beta}\, u^{\beta} \tag{8}$$

We note, in anticipation of what is to follow, that the measures of stretching and bending strain need not be precisely $E_{\alpha\beta}$ and $K_{\alpha\beta}$. Any two linear combinations from which $E_{\alpha\beta}$ and $K_{\alpha\beta}$ can be recovered might be used instead. Part of the problem in deriving a complete shell theory is to decide which two combinations of $E_{\alpha\beta}$ and $K_{\alpha\beta}$ to adopt as basic strain measures.

FORCES AND MOMENTS

As ordinarily defined, the stress measures in the theory of thin shells, namely the membrane stress tensor $N^{\alpha\beta}$ and the bending moment tensor $M^{\alpha\beta}$, must satisfy the following well-known exact system of equilibrium equations

$$N^{\alpha\beta}_{\ \ ,\alpha} + b^{\beta}_{\alpha}\, M^{\gamma\alpha}_{\ \ ,\gamma} + p^{\beta} = 0 \tag{9a}$$

$$M^{\alpha\beta}{}_{,\alpha\beta} - b_{\alpha\beta} N^{\alpha\beta} + p = 0 \tag{9b}$$

$$\varepsilon^{\alpha\beta} [N^{\alpha\beta} - b^{\beta}_{\gamma} M^{\gamma\alpha}] = 0 \tag{9c}$$

where p^{β} and p are external load intensities assumed applied at the middle surface.

It is characteristic of both approximate and exact theories of structural mechanics that they involve, as basic ingredients, a certain number of strain measures and an equal number of stress measures linked by constitutive relations. As was shown in the previous section six measures of strain are sufficient for a first order theory. But now there are apparently eight measures of stress since $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are not in general symmetric. One possible tentative remedy for this unpleasing lack of symmetry in the theory is to define modified stress measures.

$$\overline{M}^{\alpha\beta} = \frac{1}{2} (M^{\alpha\beta} + M^{\beta\alpha}) \tag{10}$$

$$\overline{N}^{\alpha\beta} = N^{\alpha\beta} - b^{\beta}_{\gamma} M^{\gamma\alpha} \tag{11}$$

which are symmetric tensors, the first by definition, the second by equation (9c). The question of the adequacy of these modified stress measures is partially resolved by the remarkable fact that the remaining equations (9a) and (9b) can be written exactly in terms of $\overline{N}^{\alpha\beta}$ and $\overline{M}^{\alpha\beta}$ as follows :

$$\left.\begin{aligned} \bar{N}^{\alpha\beta}{}_{,\alpha} + b^{\beta}_{\gamma,\alpha}\, \bar{M}^{\alpha\gamma} + 2b^{\beta}_{\gamma}\, \bar{M}^{\gamma\alpha}{}_{,\alpha} + p^{\beta} &= 0 \\ \bar{M}^{\alpha\beta}{}_{,\alpha\beta} - b_{\alpha\beta}\, b^{\beta}_{\gamma}\, \bar{M}^{\gamma\alpha} - b_{\alpha\beta}\, \bar{N}^{\alpha\beta} + p &= 0 \end{aligned}\right\} \qquad (12)$$

The idea of reducing the number of stress measures in the theory of shells from eight to six (without approximation) was first suggested by Lurie [1950] and used by Novozhilov [1951] but, as will be shown subsequently, the Lurie-Novozhilov reduction performed in lines-of-curvature coordinates is not consistent with any general tensor representation of the modified stress variables.

The temptation is now strong to adopt $\bar{N}^{\alpha\beta}$ and $\bar{M}^{\alpha\beta}$ as the basic stress measures in shell theory and to form constitutive relations between them and $E_{\alpha\beta}$ and $K_{\alpha\beta}$. But an important question presents itself: would this lead to well-defined boundary-value-problems for the determination of the configuration of the distorted shell with uniqueness of solution guaranteed?

VIRTUAL WORK, UNIQUENESS, AND MINIMUM PRINCIPLES

By means of a direct calculation we can show that the familiar three-dimensional internal-virtual-work expression $\int_V \sigma_{ij}\, \varepsilon_{ij}\, dv$ reduces exactly to the following integral over the shell middle surface

$$\int\!\!\int_A (\bar{N}^{\alpha\beta} E_{\alpha\beta} + \bar{M}^{\alpha\beta} K_{\alpha\beta})\, dA \qquad (13)$$

when the deformations are constrained by the Kirchoff assumptions.

This result is quite independent of the stress distribution. Under the same assumption the external virtual work is <u>exactly</u>

$$\int\int_A (p^\beta u_\beta + pw)\, dA + \oint_S \Big[(T^\alpha - b^\alpha_\beta t^\beta M_t)\, u_\alpha + M_n \psi_n + \Big(Q - \frac{\partial M_t}{\partial S}\Big) w\Big]\, dS \tag{14}$$

where the line integral is over the boundary S of the shell middle surface, and on S

T^α = membrane force-per-unit-length vector $(= N^{\beta\alpha} n_\beta)$

Q = transverse force per unit length $(= M^{\alpha\beta}{}_{,\alpha} n_\beta)$

M_n = normal bending moment $(= M_{\alpha\beta} n_\alpha n_\beta)$

M_t = twisting moment $(= -M^{\alpha\beta} n_\alpha t_\beta)$

ψ_n = normal rotation $(= \phi^\alpha n_\alpha)$

t^α = unit tangent vector

n^α = unit normal (in middle surface) to S.

For simplicity S is assumed smooth; otherwise additional terms enter (14) at corners. Now by virtue of the three-dimensional principle of virtual work, it follows without the need for further proof that (13) and (14) must be equal to each other whenever (a) $E_{\alpha\beta}$ and $K_{\alpha\beta}$ are derivable via (6) and (7) from displacements u^α, w, and (b) $N^{\alpha\beta}$, $M^{\alpha\beta}$, T^α, M_t, M_n and Q are the resultants of any three-dimensional stress state in equilibrium with p^α and p.

From the form of (14) one naturally defines an effective boundary membrane

force $\bar{T}^{\alpha} = T^{\alpha} - b^{\alpha}_{\beta} t^{\beta} M_t$ and an effective transverse force (as in plate theory) $\bar{Q} = Q - \frac{\partial M_t}{\partial S}$. It turns out then, that $\bar{T}^{\alpha}$, $\bar{Q}$ and M_n are expressible exactly in terms of the modified tensors $\bar{N}^{\alpha\beta}$ and $\bar{M}^{\alpha\beta}$ as follows :

$$\bar{T}^{\alpha} = (\bar{N}^{\alpha\beta} + b^{\alpha}_{\gamma} \bar{M}^{\gamma\beta} + b^{\alpha}_{\omega} t^{\omega} t_{\gamma} \bar{M}^{\gamma\beta}) n_{\beta} \tag{15}$$

$$\bar{Q} = \bar{M}^{\alpha\beta}{}_{,\alpha} n_{\beta} + \frac{\partial}{\partial S} (\bar{M}^{\alpha\beta} n_{\alpha} t_{\beta}) \tag{16}$$

$$M_n = \bar{M}^{\alpha\beta} n_{\alpha} n_{\beta} \tag{17}$$

Now, as one would suspect, the requirement (b) for the equality of (13) and (14) may be relaxed simply to the stipulation that $\bar{N}^{\alpha\beta}$, $\bar{M}^{\alpha\beta}$, p^{α}, and p satisfy the equilibrium equations (12) and that $\bar{T}^{\alpha}$, $\bar{Q}$, and M_n be defined by (15), (16), and (17). This can be verified by direct use of the divergence theorem.

We are now in a position to formulate classical types of boundary value problems with admissible boundary conditions as indicated by the form of (14). Thus we can specify either $\bar{T}^{\alpha}$ or u^{α}, M_n or ψ, $\bar{Q}$ or w; or we can impose elastic constraints by relating $\bar{T}^{\alpha}$ and u^{α}, etc. If we introduce constitutive relations between $\bar{N}^{\alpha\beta}$, $\bar{M}^{\alpha\beta}$ and $E_{\alpha\beta}$, $K_{\alpha\beta}$ which render (13) positive definite, then a unique solution of the field equations (6), (7), (12) and these constitutive relations is guaranteed (except possibly for rigid body motions). The

proof is of the usual type that exploits a virtual work principle. The question of existence remains open.

We shall not devote much attention here to the form of the required constitutive relations; suffice it to say that Novozhilov [1951] and Koiter [1960] have given compelling arguments, based on considerations of strain energy, that can be used to defend the accuracy of the uncoupled (Love) relations between $N^{\alpha\beta}$ and $E_{\alpha\beta}$, and between $M^{\alpha\beta}$ and $K_{\alpha\beta}$. We do not insist that other relations must not be used as long as they keep (13) positive definite.

With such constitutive equations adjoined to the equilibrium equations (12) and the strain-displacement equations (6) and (7) we not only have a virtual work principle (and hence uniqueness) but, in the usual fashion, we can easily formulate and prove principles of minimum potential energy and minimum complementary energy and reciprocal theorems analogous to those of three-dimensional elasticity. The use of $N^{\alpha\beta}$ and $M^{\alpha\beta}$ as stress measures has turned out to be entirely adequate because we have at this point derived a complete and satisfactory theoretical apparatus for shell analysis. But additional considerations will lead to a better theory.

ALTERNATIVE STRAIN VARIABLES

As stated previously, there was no obvious requirement for using $E_{\alpha\beta}$ and $K_{\alpha\beta}$ as the tensors describing the deformation of the shell. We shall consider alternatives, but in so doing, we shall simultaneously introduce new modified stress and moment tensors so as to leave unchanged the form and content of the virtual work expression (13), and so retain

all of the desirable features of the theory just derived.

In exploring alternative strain measures we are reluctant to discard $E_{\alpha\beta}$. Stretching of the middle surface has to do only with metric properties and has nothing to do with $K_{\alpha\beta}$. In the membrane theory of shells and in all the different bending theories of shells $E_{\alpha\beta}$ is accepted as the standard measure of the middle surface stretching strain. However, the measure of bending strain is not at all standard. We consider as alternatives to $K_{\alpha\beta}$ a class of linear combinations of $K_{\alpha\beta}$ and $E_{\alpha\beta}$ of the form

$$\tilde{K}_{\alpha\beta} = K_{\alpha\beta} - C^{\gamma\delta}_{\alpha\beta} E_{\gamma\delta} \tag{18}$$

where $C^{\gamma\delta}_{\alpha\beta}$ is required to be symmetric in α and β. For dimensional reasons, and in order that $\tilde{K}_{\alpha\beta}$ reduce to $K_{\alpha\beta}$ in the case of flat plates, $C^{\gamma\delta}_{\alpha\beta}$ will be assumed to be a homogeneous function of degree one of the shell curvature tensor. For simplicity $C^{\gamma\delta}_{\alpha\beta}$ will be assumed further to be a linear function of the components of $b_{\alpha\beta}$. Then, to within constant factors, there exist just the following four independent tensorial forms of $C^{\gamma\delta}_{\alpha\beta} E_{\gamma\delta}$ (Rivlin [1955])

$$\left.\begin{aligned} (Q_{\alpha\beta})_1 &= g_{\alpha\beta}\, b^{\omega\gamma} E_{\omega\gamma} \\ (Q_{\alpha\beta})_2 &= b^{\gamma}_{\gamma} E_{\alpha\beta} \\ (Q_{\alpha\beta})_3 &= b_{\alpha\beta} E^{\gamma}_{\gamma} \\ (Q_{\alpha\beta})_4 &= g_{\alpha\beta}\, b^{\omega}_{\omega} E^{\gamma}_{\gamma} \end{aligned}\right\} \tag{19}$$

The additional form

$$(Q_{\alpha\beta})_5 = b^{\omega}_{\alpha} E_{\omega\beta} + b^{\omega}_{\beta} E_{\alpha\omega} \tag{20}$$

which appears to belong in the list, is actually given by

$$(Q_{\alpha\beta})_5 = (Q_{\alpha\beta})_1 + (Q_{\alpha\beta})_2 + (Q_{\alpha\beta})_3 - (Q_{\alpha\beta})_4$$

according to the extended Cayley-Hamilton theorem of Rivlin [1955] .

It should be remarked that the alternative forms for $\tilde{K}_{\alpha\beta}$ given by the subtraction from $K_{\alpha\beta}$ of terms of the type in (19) are all acceptable in the sense of Koiter [1960], who shows that the error in Love's uncoupled strain energy expression (consistent with uncoupled stress-strain relations) is essentially the same no matter which of these alternatives is used. Koiter himself prefers the expression

$$\hat{K}_{\alpha\beta} = K_{\alpha\beta} - \frac{1}{2} (Q_{\alpha\beta})_5 \tag{21}$$

which, in lines-of-curvature coordinates, is the same as that of Sanders [1959], and is the one which will ultimately be preferred here. A pleasing feature of $\hat{K}_{\alpha\beta}$ remarked upon by Koiter is that it can be written neatly in the form

$$\hat{K}_{\alpha\beta} = \frac{1}{2} (\phi_{\alpha,\beta} + \phi_{\beta,\alpha}) + \frac{1}{2} (b^{\gamma}_{\alpha} \omega_{\gamma\beta} + b^{\gamma}_{\beta} \omega_{\gamma\alpha}) \tag{22}$$

in terms of the rotation vector ϕ_α and the rotation-about-the-normal tensor $\omega_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha})$. As will be seen, there are even more cogent reasons for the adoption of $\hat{K}_{\alpha\beta}$ as the standard measure of bending strain.

To retain a virtual work expression of the form (13) when $\hat{K}_{\alpha\beta}$ is adopted, $\bar{M}_{\alpha\beta}$ may be retained as the basic modified bending moment tensor, but a new modified membrane stress tensor

$$\hat{N}^{\alpha\beta} = \bar{N}^{\alpha\beta} + \frac{1}{2}(b^{\alpha}_{\gamma}\bar{M}^{\gamma\beta} + b^{\beta}_{\gamma}\bar{M}^{\alpha\gamma}) \tag{23}$$

must be introduced, giving the virtual work expression

$$\int\int_A (\hat{N}^{\alpha\beta} E_{\alpha\beta} + \bar{M}^{\alpha\beta}\hat{K}_{\alpha\beta})\, dA \tag{24}$$

The equilibrium equations in terms of $\hat{N}^{\alpha\beta}$ and $\bar{M}^{\alpha\beta}$, still exact, become

$$\hat{N}^{\alpha\beta}{}_{,\alpha} + b^{\beta}_{\gamma}\bar{M}^{\alpha\gamma}{}_{,\alpha} + \frac{1}{2}(b^{\beta}_{\gamma}\bar{M}^{\alpha\gamma} - b^{\alpha}_{\gamma}\bar{M}^{\gamma\beta})_{,\alpha} + p^{\beta} = 0 \tag{25}$$

$$\hat{N}^{\alpha\beta}{}_{,\alpha\beta} - b_{\alpha\beta}\bar{M}^{\alpha\beta} + p = 0 \tag{26}$$

Note the striking point that (26) in terms of $\hat{N}^{\alpha\beta}$ and $\bar{M}^{\alpha\beta}$ has precisely the same form as (9b) in terms of the unsymmetrical tensors $N^{\alpha\beta}$ and $M^{\alpha\beta}$. Now, anticipating the ultimate preference for $\hat{K}_{\alpha\beta}$ and $\hat{N}^{\alpha\beta}$, the general alternative version of $\tilde{K}_{\alpha\beta}$ will be written

$$\tilde{K}_{\alpha\beta} = \hat{K}_{\alpha\beta} - \sum_{i=1}^{4} c_i \, (Q_{\alpha\beta})_i \tag{27}$$

where the c_i are constants. The modified $\tilde{N}^{\alpha\beta}$ associated with $\tilde{K}^{\alpha\beta}$ that provides a virtual work density $(\tilde{N}^{\alpha\beta} E_{\alpha\beta} + M^{\alpha\beta} \tilde{K}_{\alpha\beta})$ is readily found to be

$$\tilde{N}^{\alpha\beta} = \hat{N}^{\alpha\beta} + \sum_{i=1}^{4} c_i \, (P^{\alpha\beta})_i \tag{28}$$

where the c_i are the same as those in (27), and where

$$\begin{aligned}
(P^{\alpha\beta})_1 &= b^{\alpha\beta} M^{\gamma}_{\gamma} \\
(P^{\alpha\beta})_2 &= b^{\gamma}_{\gamma} M^{\alpha\beta} \\
(P^{\alpha\beta})_3 &= g^{\alpha\beta} b_{\omega\gamma} M^{\omega\gamma} \\
(P^{\alpha\beta})_4 &= g^{\alpha\beta} b^{\omega}_{\omega} M^{\gamma}_{\gamma}
\end{aligned} \tag{29}$$

Note that the second and fourth of the forms in (19) and (29) are similar, but that $(Q_{\alpha\beta})_1$ has the structure of $(P^{\alpha\beta})_3$ and $(Q_{\alpha\beta})_3$ has that of $(P^{\alpha\beta})_1$.

We can now state categorically that the modified bending strain and membrane stress measures used by Lurie and Novozhilov in their analyses

in lines-of-curvature coordinates are not special cases of (22) and (28). The shell theory given in Novozhilov's book [1951] has a virtual work principle, and the bending strain variables are " acceptable" in Koiter's sense, but the equations can not be cast into tensor form in general coordinates for arbitrary shells.

Now we can begin to discuss the further advantages of (21) and (23) over any of the alternative forms (27) and (28) provided by all possible choices for the c_i.

THE STATIC-GEOMETRIC ANALOGY

The Russian literature on shells often refers (see, for example, Goldenveizer [1961], Novozhilov [1951], Lurie [1961]) to an interesting type of analogy between equilibrium equations (with no external loads) in terms of forces and moments on the one hand, and compatibility equations in terms of certain stretching and bending strains on the other. However, the analogy, as exhibited by Novozhilov and Lurie in lines-of-curvature coordinates involves, of course, non-tensorial force and bending strain variables. But a static-geometric analogy found independently by Sanders [1959] can be written tensorially in terms of $N^{\alpha\beta}$, $\hat{M}^{\alpha\beta}$, $E_{\alpha\beta}$, and $\hat{K}_{\alpha\beta}$ as follows.

Let

$$W^{\alpha\beta} = \hat{N}^{\alpha\beta} - i\,[\epsilon^{\alpha\omega}\,\epsilon^{\beta\gamma}\,\hat{K}_{\omega\gamma}] \qquad (30)$$

$$Z^{\alpha\beta} = \hat{M}^{\alpha\beta} + i\,[\epsilon^{\alpha\omega}\,\epsilon^{\beta\gamma}\,E_{\omega\gamma}] \qquad (31)$$

Then, in the absence of loading terms p^β and p, the real parts of the following equations are the equilibrium equations (25) and (26), while the imaginary parts are, rigorously, equations of strain compatibility:

$$\left.\begin{aligned} W^{\alpha\beta}{}_{,\alpha} + b^\beta_\gamma Z^{\gamma\alpha}{}_{,\alpha} + \frac{1}{2}\left(b^\beta_\gamma Z^{\alpha\gamma} - b^\alpha_\gamma Z^{\gamma\beta}\right)_{,\alpha} &= 0 \\ Z^{\alpha\beta}{}_{,\alpha\beta} - b_{\alpha\beta} W^{\alpha\beta} &= 0 \end{aligned}\right\} \qquad (32)$$

An immediate, useful consequence of the static-geometric analogy is that homogeneous equilibrium states can easily be generated in terms of stress functions Ω_α and ϕ simply by replacing u_α and w in the strain-displacement relations (6) and (22) by Ω_α and ϕ, and setting the resultant expressions, multiplied by $\varepsilon^{\omega\gamma} \varepsilon^{\gamma\beta}$, equal to $\bar{N}^{\omega\gamma}$ and $-\hat{N}^{\omega\gamma}$, respectively.

We now consider the question: which, if any, of the alternative versions (27), (28) for $\tilde{K}_{\alpha\beta}$ and $\tilde{N}^{\alpha\beta}$ would enjoy a static-geometric analogy? A detailed study of all the possibilities reveals that the set of admissible alternatives is reduced somewhat by this criterion, to

$$\tilde{K}_{\alpha\beta} = \hat{K}_{\alpha\beta} - c_2 (Q_{\alpha\beta})_2 - c_4 (Q_{\alpha\beta})_4 - c_1 [(Q_{\alpha\beta})_1 - (Q_{\alpha\beta})_3] \qquad (33)$$

$$\tilde{N}^{\alpha\beta} = \bar{N}^{\alpha\beta} + c_2 (P^{\alpha\beta})_2 + c_4 (P^{\alpha\beta})_4 + c_1 [(P^{\alpha\beta})_3 - (P^{\alpha\beta})_1] \qquad (34)$$

Note that the $K_{\alpha\beta}$ and $\bar{N}^{\alpha\beta}$ we first arrived at in our development are <u>not</u> members of these sets; they can <u>not</u> lead to a static-

geometric analogy, and so we reject, at this point, the previously derived theory and turn our attention to the equilibrium equations (25) and (26), and the strain-displacement equations (6) and (22). The virtual work principle, with $N^{\alpha\beta}$ and $K_{\alpha\beta}$ replaced by $\hat{N}^{\alpha\beta}$ and $\hat{K}_{\alpha\beta}$ remains intact, and the effective boundary membrane force is now given by

$$T^{\alpha} = [\hat{N}^{\alpha\beta} + \frac{1}{2}\left(b^{\alpha}_{\gamma} M^{\gamma\beta} - b^{\beta}_{\gamma} M^{\alpha\gamma}\right) + b^{\alpha}_{\omega} t^{\omega} t_{\gamma} M^{\gamma\beta}]\, n_{\beta} \qquad (35)$$

while (16) and (17) still apply. Once again, the introduction of appropriate constitutive relations provides a complete system of equations, with associated variational and reciprocal principles available. Retention of the uncoupled Love relations between $\hat{N}^{\alpha\beta}$ and $E_{\alpha\beta}$, $M^{\alpha\beta}$ and $\hat{K}_{\alpha\beta}$ still appears to be a sensible and, on the basis of Koiter's and Novozhilov's work, a justifiable procedure.

But now, before considering <u>this</u> apparatus as the definitive one, we must give some good reasons for rejecting the alternatives afforded by (33) and (34).

THE FORM OF THE EQUATIONS IN SPECIAL CASES

In contrast to the many different sets of general shell equations that have been proposed, there appears to be common agreement on the equations for symmetrical bending of shells of revolution; the classical theory of Love [1927] is generally followed in this important special case. It would clearly be desirable for a standard general theory to agree with Love's theory in this case. The presently preferred theory does so agree; in the special case considered, only the diagonal terms of $\hat{K}_{\alpha\beta}$, when

written in lines-of-curvature coordinates, are non-vanishing, and they are found to agree with the corresponding two bending strains of Love. Furthermore, none of the alternative expressions for $\tilde{K}_{\alpha\beta}$ afforded by (33) enjoy this distinction.

It might be noted, next, that curved beam theory can be regarded as a special case of cylinder theory in which the deformations are independent of the axial coordinate. In this case, with ξ^1 taken in the circumferential direction, $\hat{K}_{11}$ reduces to the measure of bending strain corresponding to the simplest curved beam theory. An alternative measure corresponding to K_{11} (equal to the linearized change of curvature) has occasionally been used but it leads to awkward energy expressions.

It may be observed, incidentally, that when lines-of-curvature coordinates are used for the general shell, the preferred expression $\hat{N}^{\alpha\beta}$ has the attractive feature that $\hat{N}^{11}$ and $\hat{N}^{22}$ are identical with the unmodified components N^{11} and N^{22} of the unsymmetrical membrane stress tensor. This is true also of the Lurie-Novozhilov modified stresses, which, in lines-of-curvature coordinates differ from $\hat{N}^{\alpha\beta}$ only in the shearing term; similarly, in the same coordinates, only the off-diagonal terms of $\hat{K}_{\alpha\beta}$ differ from the Lurie-Novozhilov components of bending strain. It appears, then, that the only one of the various characteristics that we consider desirable in a standard general shell theory that is violated by the Lurie-Novozhilov formulation is that it have a general tensor character.

CONCLUSIONS

We have arrived at a linear first order theory of elastic shells that is considered by us to be the " best" on the basis of various logical and

esthetic criteria; this theory was originally proposed by Sanders [1959] in lines-of-curvature coordinates. To sum up, the features that characterize this theory are:

(a) the equations can be written in general tensor form for arbitrary shells;

(b) the deformations are described by six strain measures, three of which are the components of the usual membrane strain tensor; the other three deviate from the components of the geometrical curvature-change tensor only by terms that are bilinear in the components of the curvature and the membrane strain tensor;

(c) the stresses are described by six stress measures that satisfy the equations of equilibrium without approximation;

(d) the theory has a principle of virtual work that is exact for displacements obeying the Kirchoff hypotheses; in conjunction with appropriate constitutive relations between the stress and strain measures, well-set boundary value problems can be formulated, and the usual minimum and reciprocal relations of structural mechanics apply;

(e) the theory contains an exact static-geometric analogy;

(f) when applied to the symmetrical bending of shells of revolution, the stress and strain measures agree with those generally used; they are consistent, too, with those of the most simple curved beam theory.

If, finally, we stipulate, for the sake of simplicity and definiteness that

(g) the stress and strain measures obey the uncoupled Love constitutive relations,then the features (a) - (g) appear to characterize uniquely

the presently preferred theory. Thus, whether or not this theory is generally adopted, it is hoped that the present study will facilitate the assessment of any given alternative theory by revealing which of the characteristics (a) - (g) it necessarily fails to embody.

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